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Refined multilayered plate elements based on Murakami zig–zag functions

Luciano Demasi *

Department of Aeronautics and Astronautics, Guggenheim Building, University of Washington, Seattle, WA 98195-2400, USA

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Abstract

This paper discusses the FEM use of Murakami zig–zag function (MZZF) in the two-dimensional modeling of multilayered plates. A literature overview is first conducted on the available works in which MZZF has been applied. MZZF is used to introduce the zig–zag effect in classical and higher order theories, which are formulated using only displacements as unknowns. MZZF is also considered to introduce the zig–zag effect in those theories, which are formulated on the basis of both displacement and transverse stress assumptions (mixed formulations). The present FEM formulation is validated by comparing the results with some available papers from the literature. A very thick plate ($a/h = 4$) is studied and the results are compared with the commercial code NASTRAN. Numerical results are presented to show both the effectiveness and limitations of MZZF in the modeling of layered plates. Linear up to forth order expansion for in-plane and out-of-plane displacements, in the thickness plate direction, has been compared. It has been concluded that MZZF consists of a valuable tool to enhance the performances of both classical and advanced theories. In particular, the conducted numerical evaluations have shown that it can be more convenient to enhance a plate theory by introducing the MZZF than refining it by adding two or three higher order terms. However, in order to well approximate local effects or study thick plates, advanced models (layer-wise or three-dimensional) are required.

Keywords: FEM; RMVT, zig–zag; Murakami; Transverse stresses; Mixed formulation; Layer-wise models; Equivalent single layer models

1. Introduction

As well documented in [1], multilayered structures show a displacement field, which exhibits a discontinuous derivative with correspondence to each layer interface. This change in slope between two adjacent layers are known as zig–zag (ZZ) effect. Transverse shear and normal strains deformations are the cause of the ZZ effect between layers that are assumed perfectly bonded together. These transverse strains cause transverse shears and normal stresses, which are continuous at each layer interface (Interlaminar Continuity (IC) for the transverse stresses). Several possibilities are known to take ZZ and IC into account in multilayered structures. Developments, see [2–7], have been made in the framework of both layer-wise models (LWM), in which the number of the unknown variables is kept dependent by the number of the layers, and Equivalent Single Layer Models (ESLM), in which the unknown variables are the same for the whole laminate. The resulting theories are often denoted as zig–zag theories (ZZT). Among the ZZT developed in the ESLM framework [8], three independent approaches are known. These were denoted in [8] as Lekhnitskii multilayered theory, Ambartsumian multilayered theory and Reissner multilayered theory, respectively. LMT and AWT describe the ZZ effect by enforcing IC via constitutive equations of the layer along with strain-displacement relations. Independent assumptions for displacement and transverse stresses are instead made in the RMT applications.

* Tel.: +1 206 685 8063; fax: +1 206 543 0217.
E-mail address: ldemasi@u.washington.edu

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In the framework of RMT applications, Murakami [9] introduced a function of the thickness coordinate able to emulate the ZZ effect. In [10] such a function was denoted as 'Murakami's zig–zag function' (MZZF). MZZF has been used in [9–14] to analyze static response of layered plates and shells in conjunction of RMT applications. Mixed finite elements for plates and shells have been developed in [15–19]. MZZF has been also applied in the framework of plate/shell theories using only displacement variables ([14] and [20]). From implementation point of view, the inclusion of MZZF in existing plate models requires the same efforts that are required *by the inclusion of an additional degree of freedom*. On the other hand, from numerical point of view, as it will be clear in this paper, inclusion of MZZF leads to significant improvements of the existing plate theories; however, these improvements are difficult to be obtained by the use of other functions which differ by MZZF. However, MZZF has been used by a few authors, see also the review made in [10]. An extensive evaluation of the use of MZZF is made in [1] using an analytical formulation.

This paper aims to contribute to establish a better understanding of the use of MZZF as a tool to introduce ZZ effects in multilayered plate structures. It extends the usage in a FEM plate model (denoted as FEM in the present work). The present paper has been organized as follows.

MZZF has been described in Section 2. Section 3 discusses the use of MZZF in the framework of plate theories, which are formulated on the basis of only displacement unknowns and Principle of Virtual Displacements (PVD) applications. This is the case of FSDT and HSDT (see [6]), and it has been denoted as *simple use of* MZZF. Section 4 presents the use of MZZF in the case of RMT applications, which are formulated in terms of both displacement and transverse stress variables by referring to Reissner's mixed variational theorem (RMVT) (see [24]). This second possibility of using MZZF has been herein denoted as *advanced use of* MZZF. Numerical results are given in Section 5. A comparison is made between analyses including MZZF and those discarding it. A comparison with the analytical results found in [1] is also made.

Full writing of governing equations as well as their solution procedures have been omitted in the present work. Interested readers are referred to the unified formulations developed for both PVD and RMVT applications which have been detailed in [10,11,19].

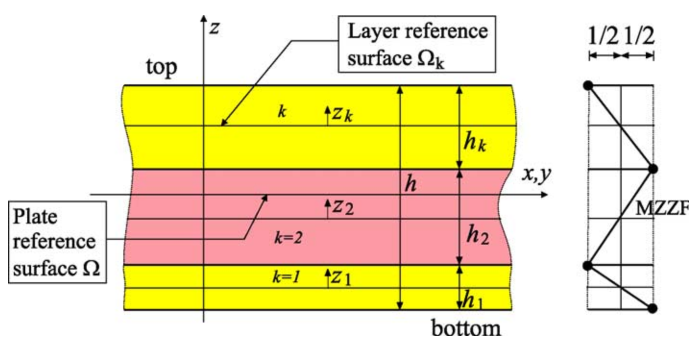

Fig. 1. Geometrical meaning of MZZF.

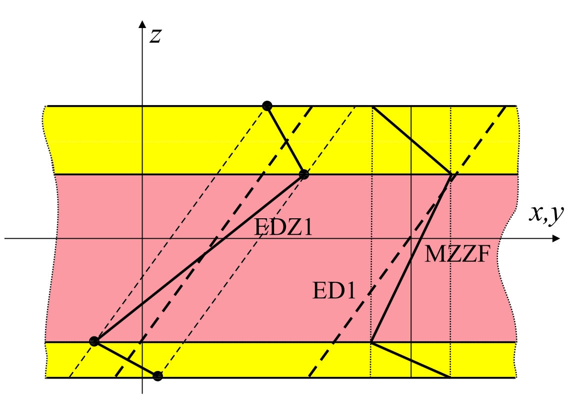

Fig. 2. Inclusion of MZZF to a linear distribution (ED1–EDZ1).

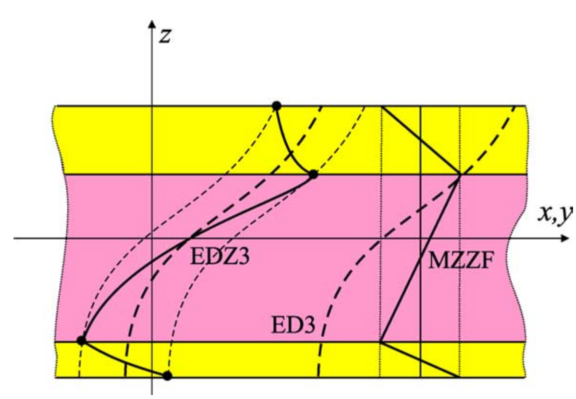

Fig. 3. Inclusion of MZZF to a higher order distribution (ED3–EDZ3).

2. The Murakami zig–zag function

Consider a multilayerd plate composed by N_l layers, which are perfectly bonded together. z is the thickness coordinate of the entire multilayerd plate while z_k is the layer thickness coordinate. The non-dimensional layer coordinate $\zeta_k = z_k/2h_k$ is further introduced (h_k is the thickness of the kth layer, as can be seen in Fig. 1). The Murakami zig–zag function $M(z)$ was defined according to the following formula [9]:

$$M(z) = (-1)^k \zeta_k \tag{1}$$

$M(z)$ has the following properties:

(1) It is a piece-wise linear function of the layer coordinate z_k.
(2) $M(z)$ has unit amplitude for the whole layers.
(3) The slope $M'(z) = \frac{dM}{dz}$ assumes opposite signs between two-adjacent layers. Its amplitude *is layer thickness dependent*.

A plot of $M(z)$ in the three-layered plate is given in Fig. 1. MZZF can be used to introduce discontinuous slopes with correspondence to layer interfaces, for any function $f(z)$. If a linear function is considered, then

$$f_1(z) = c^0 + c^1 z \tag{2}$$

where c^0 and c^1 are the amplitudes of the constant and linear terms, respectively. By adding MZZF one has

$$f_{1M}(z) = c^0 + c^1 z + c^M M(z) \tag{3}$$

where c^M is the effective amplitude of ZZ effect. f_1 and f_{1M} are compared in Fig. 2 which makes evident how $M(z)$ emulates ZZ effects. $M(z)$ can be also used in con-

Table 1
Transverse displacements $\overline{U}_z = u_z \times 100E_T h^3/(p^0 a^4)$ ($z = 0$) for a rectangular ($b = 3a$) three layered plates 0/90/0 loaded by $p_z = p^0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$ (see [21]). Evaluations of benefits of simple use of MZZF

	$\overline{U}_z$			
a/*h*	4		20	
Exact [21]	2.820		.610	
ZZT [22]	2.729		.609	
ZZT [23]	2.80		–	
Theory	ANLT [1]	FEM	ANLT [1]	FEM
Discarding MZZF				
ED1	2.051	2.058	.5633	.5634
ED2	2.035	2.141	.5668	.5660
ED3	2.627	2.670	.5955	.5957
Including MZZF				
EDZ1	2.736	2.747	.6020	.6021
EDZ2	2.719	2.786	.6043	.6046
EDZ3	2.811	2.860	.6095	.6098

Mechanical data of the lamina: $E_L/E_T = 25$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.2$ and $\nu_{LT} = \nu_{TT} = .25$. Mesh used in the present FEM analyses: 4×12 (Q9).

Table 2
Transverse displacements $\overline{U}_z = u_z \times 100E_T h^3/(p^0 a^4)$ ($z = 0$) for a rectangular ($b = 3a$) three layered plates 0/90/0 loaded by $p_z = p^0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$ (see [21]). Evaluations of benefits of advanced use of MZZF

	$\overline{U}_z$	
a/*h*	4	20
Exact [21]	2.820	.610
ZZT [22]	2.729	.609
ZZT [23]	2.80	–
Theory	FEM	
Discarding MZZF		
EMC1	2.1620	.56761
EMC2	2.1795	.56746
EMC3	2.7271	.59837
Including MZZF		
EMZC1	2.7499	.60302
EMZC2	2.7866	.60510
EMZC3	2.8632	.60977

Mechanical data of the lamina: $E_L/E_T = 25$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.2$ and $\nu_{LT} = \nu_{TT} = .25$. Mesh used in the present FEM analyses: 4×12 (Q9).

junction with higher N-order expansion $f_N(z)$ (see Fig. 3),

$$f_{NM}(z) = c^0 + c^1 z + c^2 z^2 + \cdots + c^{N-1} z^{N-1} + c^N z^N + c^M M(z) \quad (4)$$

Power of z could be replaced by any other set of polynomials of z ($P_i, i = 1, N$):

$$f_{NM}(z) = c^0 P_0(z) + c^1 P_1(z) + c^2 P_2(z) + \cdots + c^{N-1} P^{N-1}(z) + c^N P^N(z) + c^M M(z) \quad (5)$$

3. Simple use of MZZF: refinements of classical theories

Classical theories for multilayered plates, such as CLT, FSDT and HSDT, see [6,7], do not account for the ZZ effects. A possible 'simple use' of MZZF would consist of enhancing the classical models by 'simply' adding $M(z)$ in their displacement fields as explained in [1] and shown, for the FDST case, in the following equation:

$$u = u^0 + z u^1 + \boxed{(-1)^k \zeta_k u^M} \quad (6)$$

The term shown in the box is the zig–zag term. In [1], the following remarks are made:

(1) The additional degree of freedom, u^M, has been introduced with respect to FSDT. It has a meaning of displacement.
(2) The amplitude u^M is layer independent: u^M has, in fact, an intrinsic ESLM description. At a first glance this fact could appear as a strong limitation of

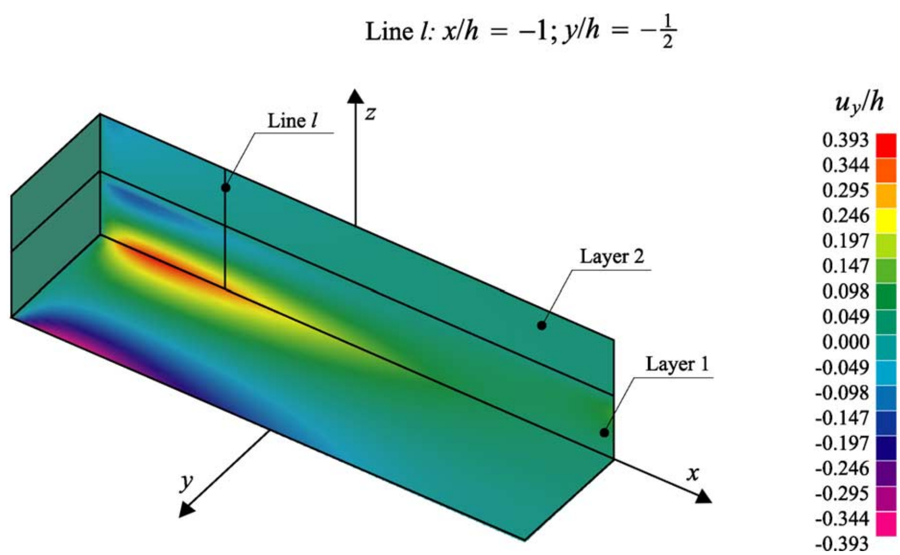

Fig. 4. Non-dimensional displacement u_y/h. Present NASTRAN results.

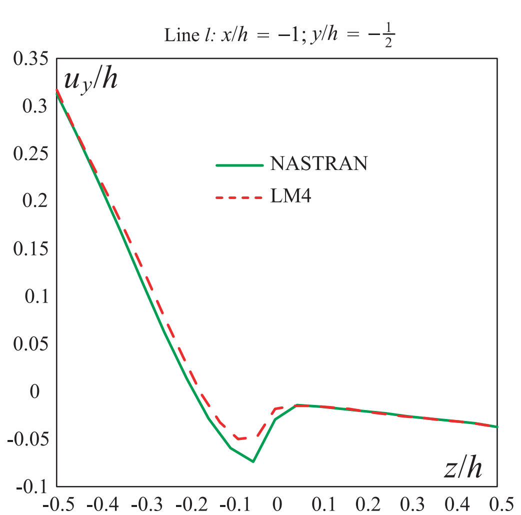

Fig. 5. Line l (see Fig. 4.): non-dimensional lateral displacement.

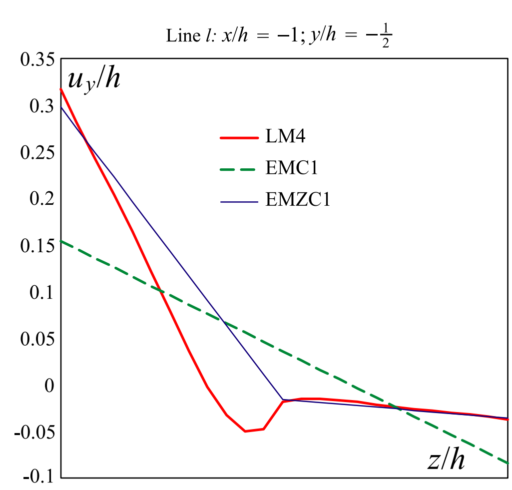

Fig. 6. Line l: advanced use of MZZF, EMC1 and EMZC1 comparison.

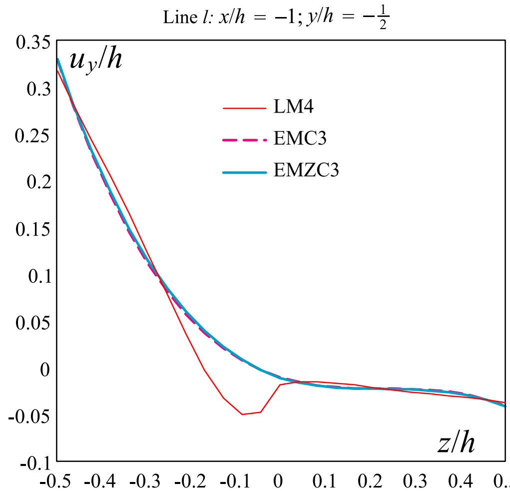

Fig. 7. Line *l*: advanced use of MZZF, EMZC3 and EMZC3 comparison.

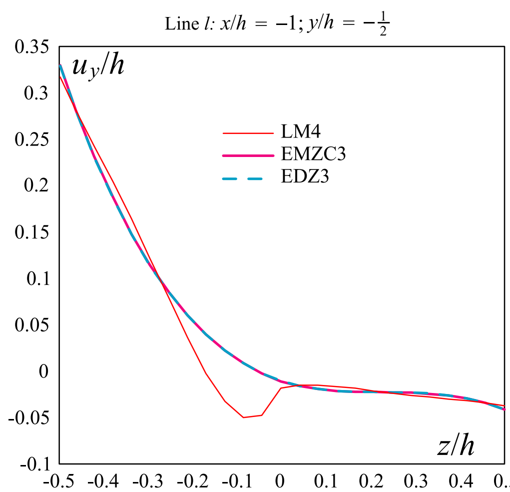

Fig. 8. Line *l*: simple and advanced use of MZZF, EDZ3 and EMZC3 comparison.

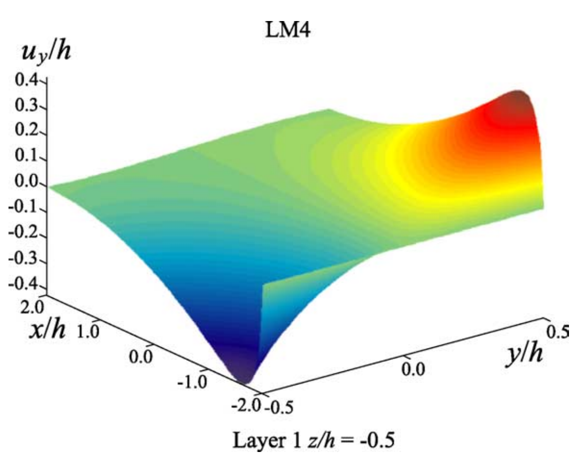

Fig. 9. LM4, u_y/h over the bottom surface ($z/h = -0.5$).

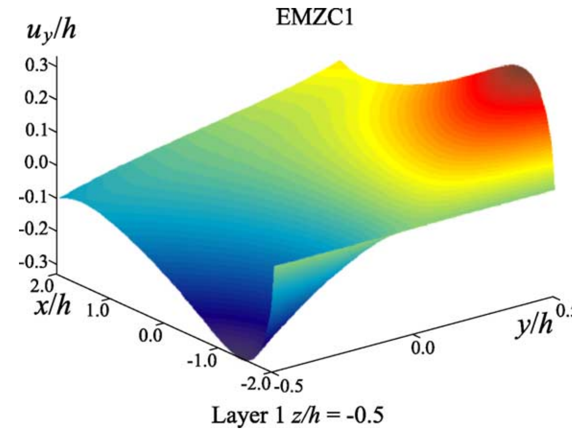

Fig. 10. EMZC1, u_y/h over the bottom surface ($z/h = -0.5$).

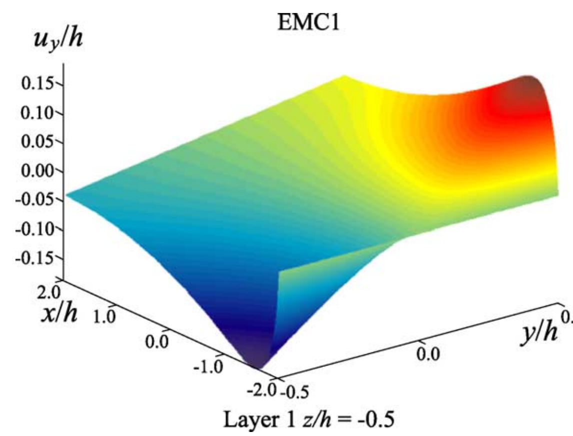

Fig. 11. EMC1, u_y/h over the bottom surface ($z/h = -0.5$).

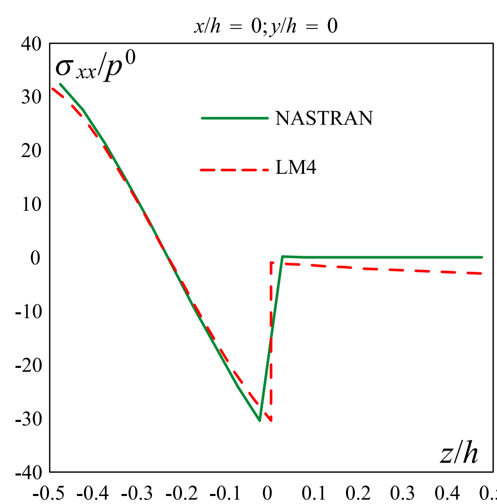

Fig. 12. Non-dimensional normal stress σ_{xx}/p^0.

MZZF. In reality, the ZZ effects have an ESLM description also in other ZZ theories, such as AWT and LMT, see [8].

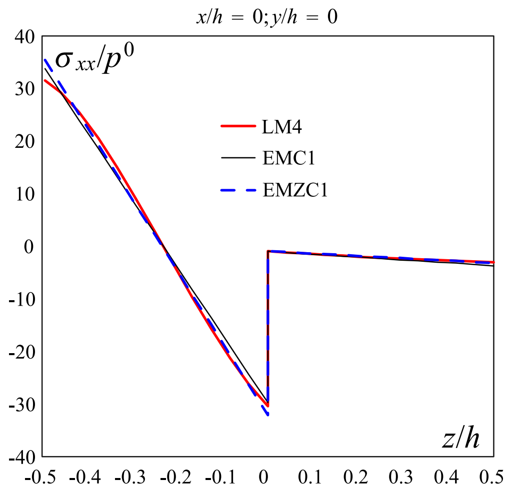

Fig. 13. Advanced use of MZZF, EMC1 and EMZC1 comparison.

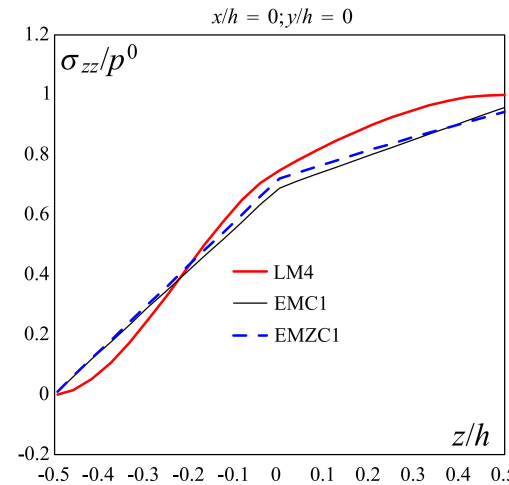

Fig. 16. Advanced use of MZZF, EMC1 and EMZC1 comparison.

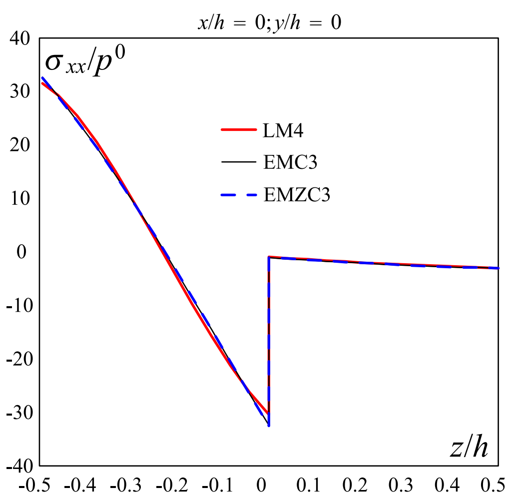

Fig. 14. Advanced use of MZZF, EMC3 and EMZC3 comparison.

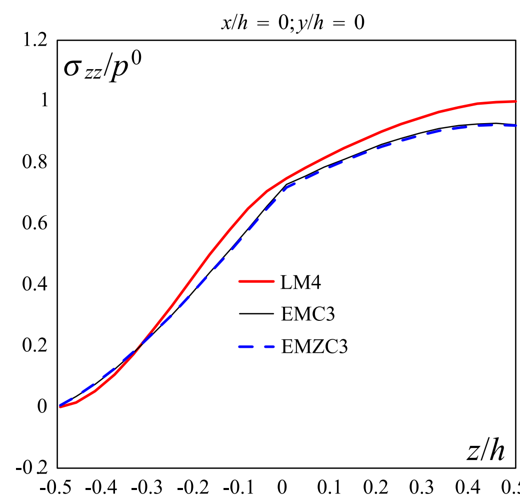

Fig. 17. Advanced use of MZZF, EMC3 and EMZC3 comparison.

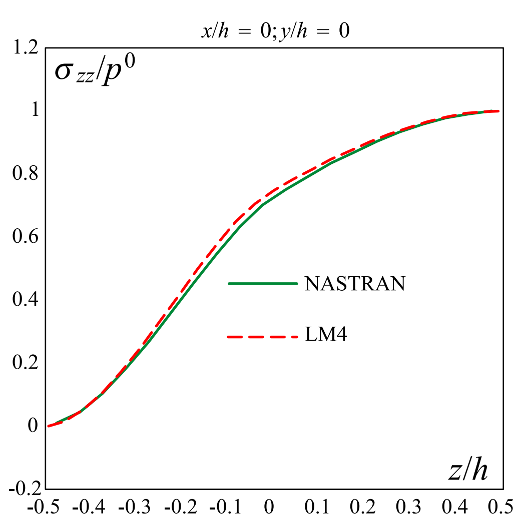

Fig. 15. Non-dimensional transverse normal stress σ_{zz}/p^0.

(3) MZZF can be used for both in-plane and out-of-plane displacement components. This is a valuable advantage of MZZF with respect to other ZZT.

3.1. Refinement of higher order theories by inclusion of ZZ effects

Consider a model of displacement, which includes transverse normal strains effects (see [19]). The displacement field can be written as

$$\begin{aligned} u_1 &= u_1^0 + zu_1^1 + \cdots + z^N u_1^N \\ u_2 &= u_2^0 + zu_2^1 + \cdots + z^N u_2^N \\ u_3 &= u_3^0 + zu_3^1 + \cdots + z^N u_2^N \end{aligned} \tag{7}$$

Such theories are called EDN (E = Equivalent single layer model; D = based on a Displacement formulation

(PVD); N = order of expansion used). EDN can be improved by adding MZZF as follows

$$u_1 = u_1^0 + zu_1^1 + \cdots + z^N u_1^N + (-1)^k \zeta_k u_1^M$$
$$u_2 = u_2^0 + zu_2^1 + \cdots + z^N u_2^N + (-1)^k \zeta_k u_2^M \quad (8)$$
$$u_3 = u_3^0 + zu_3^1 + \cdots + z^N u_2^N + (-1)^k \zeta_k u_3^M$$

Using an indicial notation

$$u_i = u_i^0 + zu_i^1 + z^2 u_i^2 + \cdots z^N u_i^N + (-1)^k \zeta_k u_i^M \quad i = 1,3 \quad (9)$$

Such theories are called using the acronym EDZN. Figs. 2 and 3 qualitatively show the ED1, EDZ1 and ED3, EDZ3 theories, respectively. The usage of MZZF is also graphically shown.

4. Advanced use of MZZF in the framework of mixed variational methods

As stated in the Introduction, the MZZF was originally introduced in the framework of RMT applications. According to the RMVT statement, two independent fields are assumed for displacements and transverse stresses. The transverse stresses are assumed independent in each layer and are interlaminarly continuous. AMT and LMT analyses introduced ZZ effect by imposing IC conditions for transverse stresses. That cannot be done by mixed approach. In this context, MZZF plays a quite fundamental role. With respect to ‘simple use of MZZF’ discussed in the previous section, the ‘advanced use of MZZF’ presented herein permits the a priori fulfillment of both ZZ and IC in multilayered plate theories. As disadvantage, an increase of the number of unknown variables is obtained. This fact can be somehow avoid by referring to the Weak Form of Hooke’s Law, presented in [10], which permits to express transverse stress variables in terms of displacement unknowns.

4.1. Stress and displacement models

The displacement models including or discarding MZZF are the same as those used in the previous section. Transverse stress fields (both shear and normal components) in each layer are conveniently written [10] in terms of Legendre polynomials according to the following expansion:

$$\sigma_\tau^k = F_{\mathrm{T}}(z_k)\sigma_{\tau\mathrm{T}}^k + F_{\mathrm{B}}(z_k)\sigma_{\tau\mathrm{B}}^k + F_2(z_k)\sigma_{\tau 2}^k + \cdots + F_{N-1}(z_k)\sigma_{\tau(N-1)}^k F_N(z_k)\sigma_{\tau N}^k \quad (10)$$

where $\sigma_{\tau\mathrm{T}}^k$ and $\sigma_{\tau\mathrm{B}}^k$ are the Top and Bottom layer values of the transverse stresses, while $\sigma_{n\tau}^k$ ($\tau = 2, N$) are the higher order terms.

$F_{\mathrm{T}}(z_k)$, $F_{\mathrm{T}}(z_k)$ and $F_r(z_k)$ are appropriate combinations of Legendre polynomials.

Subscript τ denote the three out-of-plane stress components τ = 13, 23 and 33. IC requires the fulfillment of the following relation:

$$\sigma_{\tau\mathrm{B}}^k = \sigma_{\tau\mathrm{T}}^{k-1} \quad (11)$$

As has been done for the classical theories EDN and EDZN, it is possible to define the theories EMCN and EMZCN (M = based on a mixed formulation, C = interlaminar Continuity of the transverse stresses is satisfied). In the theories EMCN the zig–zag term is not included, while in the theories EMZCN such term is used.

5. Results

5.1. Comparison with some results available in the literature

In Tables 1 and 2, the present formulation (indicated with FEM) is compared with some available results and with the analytical formulation reported in [1] and indicated with ANLT. In Table 1 the *simple use of* MZZF is reported, while in Table 2 the *advanced use* of MZZF is shown. The advantage in using MZZF in both classical and mixed FEM models is confirmed.

5.2. A proposed test case: comparison with the commercial code NASTRAN

A test case is built and the present results are compared with the commercial code NASTRAN. The analyzed structure is a cantilever plate made by two layers of equal thickness (0/90) and loaded at the top surface with a uniform pressure $p_z = p^0$. The used material is represented by the following data: $E_L/E_{\mathrm{T}} = 25, G_{LT}/E_{\mathrm{T}} = 0.5$, $G_{TT}/E_{\mathrm{T}} = 0.2$ and $\nu_{LT} = \nu_{TT} = .25$. The length is indicated with a, while the width and thickness are indicated with b and h, respectively. In the examined case, the following geometric ratios are used: $a/h = 4$

Table 3
Non-dimensional transverse displacements u_z/h at $x/h = 2$ and $y/h = z/h = 0$

NASTRAN	LM4	EMC1	EMZC1	EMC3	EMZC3
106.398	109.80	105.10	107.10	108.41	108.37
(214326)	(45927)	(10206)	(15309)	(20412)	(25515)

Comparison between FEM and NASTRAN (inside the parentheses the number of DOF are reported).

and $b/h = 1$. Clearly, in such structure the three-dimensional effects are important, and Equivalent Single Layer Models cannot describe the displacement and stress fields properly. However, the usage of MZZF improves the results, as will be shown below.

In Fig. 4 the non-dimensional displacement u_y/h, calculated using NASTRAN, [1] is shown. Fig. 4 also reports the position of the line l, where the non-dimensional displacement u_y/h is diagrammed along z/h in Fig. 5. In particular, in Fig. 5, the present FEM results [2] are indicated with LM4 and compared with the results obtained using NASTRAN. As can be seen, the layer-wise theory LM4 (see [19] for more details of such layer-wise theory) shows very good agreement with NASTRAN. It should be noticed that this result is very relevant, since the structure is very thick and the non-dimensional displacement u_y/h is, clearly, a local effect: with the two-dimensional FEM element LM4, it is possible to capture local effects very well. But the goal of the present paper is to demonstrate the utility of the MZZF. Therefore, the previous result is only shown as a reference for future papers.

In Fig. 6, it is possible to see the quality of the Equivalent Single Layer Models: they are not very accurate, [3] but the improvement obtained by adding the MZZF is great.

In Fig. 7, a similar comparison (as has been done in Fig. 6) is performed. The theories EMC3 and EMZC3 are compared. As concluded in [1] and confirmed here, the usage of MZZF is effective for low order of the expansion (see Fig. 6).

Fig. 8, shows that classical and advanced usage of MZZF produce similar results.

In Figs. 9–11 the non-dimensional displacement u_y/h over the plate at $z/h = -0.5$ is reported. Comparing Figs. 9–11 the advantage of using MZZF is clear: the displacements are better approximated when the MZZF is adopted.

Figs. 12–14 report the non-dimensional normal stress σ_{xx}/p^0 ($x/h = 0$, $y/h = 0$ and z/h varies). It can be seen that the non-dimensional normal stress σ_{xx}/p^0 is very well approximated even without the zig–zag term.

Figs. 15–17 report the non-dimensional transverse normal stress σ_{zz}/p^0 ($x/h = 0$, $y/h = 0$ and z/h varies). No significant difference is found in using or discarding the zig–zag effect.

Finally, in Table 3, the number of DOF used in the theories and NASTRAN is reported.

[1] The used NASTRAN elements are EXA (with 8 nodes) and a mesh $80 \times 20 \times 20$ has been adopted.

[2] The used mesh is 40×10 (Q9).

[3] The structure is very thick and a local effect is considered; therefore, the fact that the Equivalent Single Layer Models are not very accurate is obvious.

6. Conclusions

The numerical performance of the inclusion of Murakami's zig–zag function using a FEM analysis has been investigated for multilayered plate theories. MZZF has been applied in both classical theories (based on PVD) and advanced theories (based on RMVT). The following conclusions can be made:

- The present FEM formulation has shown very good correlation with respect to the corresponding analytical formulation (see [1]).
- The inclusion of MZZF is very effective, especially if the order of the expansion along the thickness is low.
- In order to capture local effects, the inclusion of MZZF is not enough, and advanced layer-wise theories or three-dimensional models are required.

References

[1] Carrera E. On the use of Murakami's zig–zag function in the modeling of layered plates and Shells. Compos Struct 2004;82: 541–54.

[2] Lekhnitskii SG. Strength calculation of composite beams. Vestnik inzhen i tekhnikov 1935;9.

[3] Lekhnitskii SG. Anisotropic plates, 2nd ed. 1968. Translated from the 2nd Russian ed. by S W Tsai, Cheron, Bordon and Breach.

[4] Ambartsumian SA. Theory of anisotropic shells, Fizmatzig, Moskwa; 1961. Translated from Russian, NASA TTF-118, (1964).

[5] Ambartsumian SA. Theory of anisotropic plates, 1969. Translated from Russian by T. Cheron and Edited by JE Ashton Tech Pub Co.

[6] Reddy JN. Mechanics of laminated composite plates, theory and analysis. CRC Press; 1997.

[7] Carrera E. Theories and finite elements for multilayered anisotropic, composite plates and shells. Archiv Comput Methods Eng State of the art Reviews 2002;9:87–140.

[8] Carrera E. Historical review of zig–zag theories for multilayered plates and shells. Appl Mech Rev 2003;56:287–308.

[9] Murakami H. Laminated composite plate theory with improved in-plane response. J Appl Mech 1986;53:661–6.

[10] Carrera E. Developments, ideas and evaluations based upon the Reissner's mixed theorem in the modeling of multilayered plates and shells. Appl Mech Rev 2001;54:301–29.

[11] Carrera E. Theories and finite elements for multilayered plates and shells: a unified compact formulation with numerical assessment and benchmarking. Archiv Comput Methods Eng 2003; 10(3).

[12] Toledano A, Murakami H. A high-order laminated plate theory with improved in-plane responses. Int J Solids Struct 1987;23: 111–31.

[13] Jing H, Tzeng KG. Refined shear deformation theory of laminated shells. Am Inst Aeronautics Astronautics J 1993;31: 765–73.

[14] Carrera E. A study of transverse normal stress effects on vibration of multilayered plates and shells. J Sound Vib 1999;225:803–29.

[15] Rao KM, Meyer-Piening HR. Analysis of thick laminated anisotropic composites plates by the finite element method. Compos Struct 1990;15:185–213.

[16] Brank B, Carrera E. Multilayered shell finite element with interlaminar continuous shear stresses: a refinement of the

Reissner–Mindlin formulation. Int J Numer Methods Eng 2000; 48:843–74.
[17] Carrera E, Parisch H. Evaluation of geometrical nonlinear effects of thin and moderately thick multilayered composite shells. Compos Struct 1998;40:11–24.
[18] Auricchio F, Sacco E. Refined first-order shear deformation theory models for composite laminates. J Appl Mech 2003;70: 381–90.
[19] Carrera E, Demasi L. Multilayered finite plate element based on Reissner mixed variational theorem. Part I: theory and Part II: numerical analysis. Int J Numer Methods Eng 2002;55:191–231 [pp. 253–291].
[20] Bhaskar K, Varadan TK. A higher-order theory for bending analysis of laminated shells of revolution. Comput Struct 1991;40:815–9.
[21] Pagano NJ. Exact solutions for composite laminates in cylindrical bending. J Compos Mater 1969;3:398–411.
[22] Idlbi A, Karama M, Touratier M. Comparison of various laminated plate theories. Compos Struct 1997;37:173–84.
[23] Ren JG. A new theory for laminated plates. Compos Sci Technol 1986;26:225–39.
[24] Reissner E. On a certain mixed variational theory and a proposed applications. Int J Num Methods Eng 1984;20:1366–8.